Planar Spin Network Coherent States

I. General Properties

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Abstract

This paper is the first of a series of three which construct coherent states for spin networks with planar symmetry. Constructing such states is not straightforward. After gauge-fixing, the full SU(2) symmetry is broken to U(1); but one cannot simply use the U(1) limit of SU(2) coherent states, because the planar states exhibit an unexpected O(3) symmetry arising from the closed loop character of the transverse directions. This paper uses an intuitive, rather than mathematically rigorous approach to construct a candidate set of O(3) symmetric coherent states. However, paper 2 of the series then demonstrates explicitly that the proposed coherent states are approximate eigenvectors of the holonomy and momentum operators $\tilde{\mathrm{E}}^i_I$ (as expected for coherent states), up to small correction terms. These coherent states are superpositions of holonomies which obey the residual U(1) gauge symmetry only on average; that is, some holonomies in the superposition violate the symmetry, although the U(1) quantum numbers of these holonomies are peaked at values which obey the symmetry. An appendix discusses the closely related case of cylindrical symmetry.

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I Introduction

This paper is based on the canonical, spin network formulation of quantum gravity. This is an approach built on a connection, rather than metric formulation of gravity [1, 2]. The connection is real [3]. Basic variables are a densitized inverse triad and an exponentiated connection, i. e. a holonomy. Other operators of the theory are constructed by commuting holonomy with volume or Euclidean Hamiltonian [4]. The space on which the holonomy exists is a one-dimensional network of edges meeting at vertices, rather than a continuous manifold [5, 6].

At present there is no empirical confirmation for the existence of this microscopic spin network structure, although in the future it may be possible to detect Planck-scale modifications to elementary particle decay rates, or modifications to light propagation over cosmological distances [7, 8, 9, 10]. For now, the approach must be checked by undertaking calculations which confirm the internal consistency of the formulation, or confirm consistency with established principles. For example, area and volume operators for the theory possess a discrete spectrum [11], in qualitative agreement with the conclusions from thought experiments that lengths smaller than a Planck length are not measurable [12]. Cosmological calculations, extended back to the big bang, yield a finite result [13]. Black hole calculations predict area $\propto$ entropy and yield a formula for the entropy [14, 15].

One topic which requires further investigation is the classical limit of the spin network approach. The systems studied so far (black holes, homogeneous cosmologies) have reasonable classical limits, but they are so highly symmetric that they cannot propagate gravitational waves. In order for the system to propagate waves, the spin network Hamiltonian must be nonlocal: a single term in the Hamiltonian must be able to change the spin network at two or more neighboring vertices [16]. Once the Hamiltonian is made nonlocal, it is no longer obvious that the constraints have the correct commutation relations, in the limit of fields varying slowly over many spin network vertices [17]. I.e., it is not obvious that the theory possesses general coordinate invariance in the classical limit.

Several approaches use the path integral, or "spin foam" approach to study the classical limit, rather than the canonical, spin

network approach. Aleschi and Rovelli calculate the gravitational Green's function, then check for a correct long-range behavior [18]. This approach puts limits on allowable spin foam vertices. Ambjorn, Jurkiewicz, and Loll put the system in a heat bath and solve numerically for the ground state, to see if the geometry is reasonable [19]. This work puts limits on the topology of admissible paths, and favors spin networks which are causal. Markopoulou, and also Oeckl, have studied the application of the renormalization group to gravity theory[20, 21].

Thiemann and Winkler develop another approach to the classical limit: within the canonical approach, construct coherent states which can be used to study the theory approximately, in the classical limit [22, 23, 24]. This is the approach followed in the present paper, which constructs coherent states for a space possessing two commuting spacelike Killing vectors which may be written $\partial/\partial x$ and $\partial/\partial y$. This planar symmetry is the simplest which allows propagation of gravitational waves, and therefore requires a nonlocal Hamiltonian.

For this case one cannot use the coherent states constructed by Thiemann and Winkler. In one sense these states are too general. They possess the full SU(2) local gauge invariance, whereas planar symmetry allows the gauge to be fixed, until only U(1) gauge rotations around the Z axis survive [25]. (Lower case coordinates x,y,z,t refer to coordinates on the global manifold; upper case coordinates X,Y,Z,T refer to coordinates in local Lorentz frames.) In another sense, the Thiemann Winkler states are not quite general enough, because the planar states must possess an O(3) symmetry which is not a limit of the SU(2) gauge symmetry [26, 27].

This is the first of a series of three papers which construct coherent states for the planar case [28, 29]. In any discussion of coherent states, there is always a danger that one can get buried in detail. The discussion can degenerate into a collection of statements, with each one-sentence statement followed by a two-page proof. In an effort to avoid this, I have adopted a three-paper approach which allows readers to take in as much or as little detail as suits their purposes, as follows.

The present paper gives a general overview and relatively few details. I will refer to this paper as paper 1. Section II of the present paper describes a suitable Hilbert space for plane waves, and discusses the need for the O(3) symmetry. This section is a

summary of an earlier paper by the author [26], and is included to make the present paper reasonably self-contained.

Section III constructs coherent states for the planar case. Thiemann and Winkler derive their SU(2) coherent states in two ways. They use results of Hall on the mathematics of coherent states on group manifolds [30]; and they show that Hall's methods parallel a procedure in standard quantum mechanics for turning a free particle wavefunction into a minimal uncertainty wave packet. The latter derivation is perhaps easier to grasp intuitively, and I have followed it in section III.

I follow an intuitive approach when constructing coherent states in the present paper. Therefore I must show explicitly that the states have the properties expected of coherent states. This is done in paper 2, the second paper in the series [28]. This paper shows that the coherent states are approximate eigenfunctions of the operators which will be used to construct the Hamiltonian:

$$\mathrm{O} \mid \text{coherent state}\rangle = < \mathrm{O} > \mid \text{coherent state}\rangle + \mathrm{SC},$$

where the operator O is either a densitized inverse triad $\tilde{\mathrm{E}}^I_i$ or a spin one-half holonomy. The main body of paper 2 derives the leading contribution to the matrix element (the c-number function $< O >$) and gives qualitative estimates for the size of the small correction (SC) terms. Quantitative estimates of the SC terms are worked out in the appendices to paper 2.

Some readers may not wish to contemplate the full detail of paper 2, or they may prefer a short introduction before reading the details. For those readers, section IV of the present paper gives an overview of the main steps in the calculation of matrix elements. The focus is on the structure of the calculation (which steps are done in what order, and why). This section contains a little detail, but not yet full detail.

Cylindrical symmetry is very closely related to planar symmetry. Both metrics admit two commuting, spacelike Killing vectors. Appendix A discusses the modifications needed to handle cylindrical rather than planar symmetry.

Constructing a spin network Hamiltonian is a little like attempting to navigate through a darkened room full of furniture. In the case of the room, it is virtually impossible to proceed in any direction without barking one's shins against an obstacle, which requires

a choice of new direction. In the case of the Hamiltonian, one cannot proceed far without bumping into some ambiguity, which again requires a choice.

The present series of papers might appear to be only an explication of the properties of coherent states, independent of any choice of Hamiltonian; however, one chooses to emphasize certain properties of coherent states, because these are the properties one expects will be relevant later when calculating the Hamiltonian. There is at least an implicit choice of Hamiltonian involved in a paper of this type. I will try to make explicit relevant ambiguities and choices.

One such choice is the assumption, made above, that the Hamiltonian is constructed from spin 1/2 holonomies in the fundamental representation of SU(2). This seems to contradict results from the spin foam approach to quantum gravity. In this approach, one starts with an initial spin network living in n space dimensions, then propagates it forward in time by gluing appropriate n+1 dimensional simplices to the initial network. For a concrete, relatively simple example, consider the Euclidean version of 2+1 gravity, with all edges labeled by SU(2) spins. The spin network version of this theory is given by Rovelli [31]. The edges which are added at each time step can have arbitrary spin; there is no limitation to new edges with spin 1/2 or to new spins which differ from old by $j \rightarrow j \pm 1/2$. The evolution does not seem to be generated by a Hamiltonian which singles out spin 1/2.

Ideally, this particular shin-bark would dictate a unique change of direction: given the spin foam unitary evolution operator for 2+1 gravity, discover the Hamiltonian which generates the evolution. Presumably, the Hamiltonian would contain holonomies in representations of higher spin than 1/2. The natural choice of unitary evolution operator for this theory is a sum over a set of basic spin rearrangements (the so-called Pachner moves); but the Hamiltonian which generates this operator has not been found [32]. (If the Hamiltonian itself is made into a sum of Pachner moves, it does not behave like a constraint; it does not annihilate physical states.) Spin foam theories which are more sophisticated than 2+1 gravity can be even harder to reconcile with the canonical approach [33].

At the present time, there seems to be a disconnect between the spin foam and canonical approaches. The spin foam approach can construct a convincing unitary evolution operator, but no Hamilto-

nian constraint. The canonical approach has a constraint, but the formalism is frozen in time. In this situation, it is necessary to make a choice. I remain on the canonical side of the canonical-covariant divide and assume the holonomies are spin 1/2. By moving in this direction, of course, I may end up merely knocking over a lot more furniture.

Although spin 1/2 is the simplest choice, I should report that Gaul and Rovelli have investigated Hamiltonians involving higher spin [34]. They find there is no problem in principle with using the higher holonomies.

A third paper in the series contains some speculations about the form of the Hamiltonian [29]. Although paper 3 does use some results from papers 1 and 2, it relies primarily on dimensional analysis arguments. In logical terms it is independent of the first two papers and should be evaluated independently.

The coherent states constructed in this paper are not exact eigenstates of the residual U(1). Exact eigenstates could be obtained by angle-averaging the coherent states which will be constructed in section III. But these coherent states are already quite complicated, even before an angle average. It seems simplest to test for general covariance and gravitational wave propagation, initially using states which obey U(1) only on average; later, one can refine the calculation. For studies of angle-averaged states which obey SU(2) exactly, see reference [35].

II The Planar Hilbert Space

In reference [26] I constructed a kinematic basis for the planar case, and for completeness I review the highlights of that construction here. Call the direction of propagation the z direction. I will use lower case x, y, z, a, b,$\cdots$ for global coordinates, and upper case X, Y, Z, A, B,$\cdots$ for local SU(2) indices.

Gauge fixing may be used to simplify the $\tilde{\mathrm{E}}$ and connection fields [25]. The off-diagonal elements $\tilde{\mathrm{E}}^a_Z$ and $\tilde{\mathrm{E}}^z_A$, with a = x,y and A = X,Y, can be gauged to zero; similarly, A^a_Z and A^z_A may be set to zero. This means that the holonomies along the longitudinal z direction are quite simple, involving only the rotation generator S_Z for rotations around Z, while the transverse holonomies (those along the x and y directions of the spin network) involve S_X, S_Y and are

rotations in the X,Y plane. Each transverse holonomy therefore has an axis of rotation of the general form

$$\hat{\mathrm{m}} = (\cos\phi, \sin\phi, 0), \tag{1}$$

for some angle ϕ. (More precisely, there is one holonomy for each transverse direction and one ϕ for each transverse direction, ϕ_x and ϕ_y. Since the two directions are treated equally, I will discuss only the x holonomies, and will suppress the subscript x for now.) The corresponding spin 1/2 holonomy is

$$\begin{aligned} h^{(1/2)} &= \exp[i\hat{m}\cdot\vec{\sigma}\,\theta/2] \\ &= \exp[-i\sigma_Z(\phi-\pi/2)/2]\exp[i\sigma_Y\theta/2]\exp[+i\sigma_Z(\phi-\pi/2)/2] \\ &= h^{(1/2)}(-\phi+\pi/2, \theta, \phi-\pi/2) \end{aligned} \tag{2}$$

On the last two lines I have written the usual Euler angle decomposition for this rotation.

Given the above, the natural basis for the transverse holonomies might seem to be the set of rotation matrices

$$\mathrm{h}^{(\mathrm{j})}(-\phi+\pi/2, \theta, \phi-\pi/2),$$

where j is the highest weight obtained by multiplying together 2j $h^{(1/2)}$ matrices. However, this basis is not convenient because it has complicated behavior under the action of the $\tilde{\mathrm{E}}$. To obtain objects with simpler behavior, note that the matrix $h^{(1/2)}$ has only three independent components. (Because the matrix is special, a rotation around an XY axis, the two diagonal elements are equal: $h_{++} = h_{--}$.) Group the three independent components as follows:

$$\begin{aligned} &(V_X, V_Y, V_Z) \\ &\quad = (1/2)(h^{(1/2)}_{+-} - h^{(1/2)}_{-+}, ih^{(1/2)}_{+-} + ih^{(1/2)}_{-+}, h^{(1/2)}_{++} + h^{(1/2)}_{--}) \\ &\quad = (\sin(\theta/2)\cos(\phi-\pi/2), \sin(\theta/2)\sin(\phi-\pi/2), \cos(\theta/2)); \end{aligned} \tag{3}$$

the inverse relation is

$$\mathbf{h}^{(1/2)} = \mathbf{1}\mathrm{V}_3 + \mathrm{i}\tilde{\mathrm{V}}\cdot(\boldsymbol{\tilde{\sigma}}\times\hat{\mathrm{Z}}), \tag{4}$$

where boldface denotes a 2x2 matrix. Under action of an $\tilde{\mathrm{E}}$, the three V components rotate infinitesimally like the components of a three-vector.

$$\begin{aligned}
\tilde{\mathrm{E}}^x_A h^{1/2} &= \tilde{\mathrm{E}}^x_A \exp[i \int \mathrm{A}^B_x S_B dx] \\
&= [S_A h^{1/2} + h^{1/2} S_A)](\gamma\kappa/2) \\
&\Leftrightarrow \\
\tilde{\mathrm{E}}^x_A V_M &= V_N \langle N \mid S_A \mid M \rangle (\gamma\kappa/2). \qquad (5)
\end{aligned}$$

Subscripts A,B = X, Y only. On the second line there is a generator S_A on both sides of the holonomy, because the holonomy in a transverse direction is a closed loop, beginning and ending at the same vertex; therefore the $\tilde{\mathrm{E}}$ (which acts at the vertex) grasps the holonomy at both ends. This grasp-at-both-ends feature, characteristic of closed loops, gives rise to the O(3) symmetry. That symmetry may be exhibited by introducing the linear combinations V. From the last line, above, the grasp is just an infinitesimal O(3) rotation, when expressed in terms of the V's.

Some technical points: In eq. (5) I have omitted the delta functions coming from the [$\tilde{\mathrm{E}}$,A] commutators, because the delta functions are always canceled by the area and line integrals associated with $\tilde{\mathrm{E}}^x_A$ and A^B_x. (I always suppress the area integration associated with each $\tilde{\mathrm{E}}$.) The overall factor 1/2 is a relic of the integrals over the deltas, 1/2 because the deltas occur at the endpoints of the edge integration. $\gamma\kappa$ is the product of Immirzi parameter times 8πG; $\hbar = c = 1$.

I now return to the main point: because the V's transform more simply than the h's under the action of $\tilde{\mathrm{E}}$, one should multiply together V's rather than h's to get a basis set of holonomies. Since the grasps generate an O(3), and the V's form a vector under the action of this O(3), the products of V's generate the usual spherical harmonics of O(3). In particular, components of V itself form the simplest spherical harmonic of O(3),

$$\begin{aligned}
(V_\pm, V_0) &= (\mp \sin(\theta/2) \exp[\pm i(\phi - \pi/2)]/\sqrt{2}, \cos(\theta/2)); \\
V_M(h) &= \sqrt{4\pi/(2L+1)} Y_{L=1,M}(\theta/2, \phi - \pi/2). \qquad (6)
\end{aligned}$$

Polynomials of degree L in components of V can be expanded in

a series of the higher harmonics Y_{LM}, which therefore form a basis. (More precisely, there are two bases, $Y_{Lx,Mx}$ and $Y_{Ly,My}$ for holonomies along the x and y directions respectively.) These harmonics transform simply under the action of $\tilde{\mathrm{E}}$:

$$(\gamma\kappa/2)^{-1}\tilde{\mathrm{E}}^x_{\pm}\mathrm{Y}_{\mathrm{LM}} = \Sigma_{\mathrm{N}}\mathrm{Y}_{\mathrm{LN}}\langle \mathrm{L}, \mathrm{N} \mid \mathrm{S}_{\pm} \mid \mathrm{L}, \mathrm{M}\rangle, \tag{7}$$

where $Y_{LM} = Y_{LM}(\theta/2, \phi - \pi/2)$. The unconventional half-angle reminds us of the origin of these objects in a holonomy $h^{1/2}$ depending on half-angles.

The Y's are known to be proportional to matrix elements of rotations,

$$\sqrt{4\pi/(2\mathrm{L}+1)}\mathrm{Y}_{\mathrm{LM}} = \mathrm{D}^{(\mathrm{L})}_{0\mathrm{M}}(-\phi + \pi/2, \theta/2, \phi - \pi/2). \tag{8}$$

Therefore eq. (7) is also correct if D's are substituted for Y's. I prefer D's to Y's in what follows, because use of D's (will require awkward factors of $\sqrt{4\pi/(2L+1)}$ in initial formulas, but) ultimately will result in fewer factors $\sqrt{4\pi/(2L+1)}$. The basic vector $V(h)_M$ is also simply related to a D:

$$\begin{aligned} \hat{V}(h)_M &= \sqrt{4\pi/(2L+1)}Y_{L=1,M}(\theta/2, \phi - \pi/2) \\ &= D^{(1)}_{0M}(-\phi + \pi/2, \theta/2, \phi - \pi/2). \end{aligned} \tag{9}$$

From now on I will use a hat notation to emphasize that $\hat{V}$ is a unit vector.

The planar calculation involves three groups: SU(2), because the holonomies in the Hamiltonian are rotation matrices in SU(2); O(3), because the action of the grasps generates an O(3) group; and U(1), because the usual SU(2) gauge invariance is broken to U(1) by the gauge fixing. It is worth taking a minute to contemplate when to use which group.

Presumably the Hamiltonian will be constructed using holonomies $h^{(1/2)}$ in the fundamental representation of SU(2). The Euler decomposition of $h^{(1/2)}$, eq. (2), shows $h^{(1/2)}$ as depending on the full angle θ, but of course the actual matrix elements of $h^{(1/2)}$ contain half-angles $\sin(\theta/2), \cos(\theta/2)$. When these matrix elements are rearranged to form a vector $D^{(1)}(h)$ in O(3), the matrix elements of $D^{(1)}(h)$ (and $D^{(L)}(h)$, eq. (8)) inherit this dependence on half-angles.

I am therefore using the same notation h for an SU(2) matrix and an O(3) matrix; yet the SU(2) and O(3) matrices have different dependence on the angle θ. Hopefully this will cause no confusion, because of the following circumstance. If the matrix has a superscript (1/2), as $h^{(1/2)}$, then it is in SU(2), and the Euler decomposition involves the full angle; if the matrix has a superscript other than (1/2), then it is in O(3), and the Euler decomposition involves a half angle.

As mentioned above, presumably the Hamiltonian will contain holonomies in the fundamental representation of SU(2), $h^{(1/2)}$. The coherent states, on the other hand, will be sums of representations of O(3). Therefore when the Hamiltonian acts on the state, for calculational purposes one should replace the SU(2) $h^{(1/2)}$ matrices by the linear combinations $D^{(1)}(h)$ which transform simply under O(3):

$$\begin{aligned}
& h^{(1/2)}((-\phi+\pi/2,\theta,\phi-\pi/2) \mid \mathrm{coh}\rangle \\
\rightarrow\ & D^{(1)}_{0M}(-\phi+\pi/2,\theta/2,\phi-\pi/2) \mid \mathrm{coh}\rangle \\
=\ & < D^{(1)}_{0M}(-\phi+\pi/2,\theta/2,\phi-\pi/2) >\mid \mathrm{coh}\rangle + SC,
\end{aligned} \tag{10}$$

where the brackets denote the peak, or average value, and SC denotes small corrections (typically down by $1/\sqrt{L}$). The details of the SU(2) $\rightarrow$ O(3) replacement are given by eq. (3). Once the peak values of the $D(1)(h)$ are known, one can solve for the peak values of the $h^{(1/2)}$ using the inverse equations, eq. (4).

Now consider the relation between O(3) and the residual gauge invariance U(1). Most of the O(3) rotations have nothing to do with SU(2) gauge invariance. The exceptions are rotations around Z, which are identical to the residual U(1) gauge invariance. Proof: because of the gauge fixing, the axis of rotation for the matrices $h^{(1/2)}$ must remain in the XY plane. This reduces the gauge rotations from SU(2) to the set of U(1) rotations around the Z axis. Under an infinitesimal Z rotation θ is unchanged, while $\phi \rightarrow \phi + \delta\phi$. The corresponding changes in the matrices $h^{(1/2)}$ and the O(3) harmonics are

$$\begin{aligned}
h^{(1/2)}_{mn} &\rightarrow h^{(1/2)}_{mn} \exp[i(n-m)\delta\phi]; \\
D^{(L)}_{0M} &\rightarrow D^{(L)}_{0M} \exp[iM\delta\phi].
\end{aligned}$$

The last line is identical to the result of an O(3) rotation around the Z axis. □

It is straightforward to work out the consequences of U(1) gauge invariance. Each vertex has one x holonomy, one y holonomy, and two z holonomies (one entering or initial, one leaving or final). The total change in phase of the vertex under a Z rotation is therefore

$$\exp[\mathrm{i}\delta\beta(\mathrm{M_x} + \mathrm{M_y} + \mathrm{M_{Zf}} - \mathrm{M_{Zi}})],$$

where the M_Z come from the Z holonomies, which go as

$$\exp\,[\mathrm{i}\,\mathrm{M_Z} \int \mathrm{A}_z^Z \mathrm{dz}].$$

U(1) invariance demands that the quantity

$$(\mathrm{M_x} + \mathrm{M_y} + \mathrm{M_{Zf}} - \mathrm{M_{Zi}})$$

vanish, for each vertex.

The coherent states constructed here will not have unique values for M_x and M_y. The states are superpositions of $D^{(L_i)}_{0M_i}(h_i)$ matrices (i = x,y); and the superposition contains a range of values M_i. The superposition is sharply peaked at central values of the M_i, however, so that all values of $M_x + M_y$ in the packet are very close to the average value

$$< \mathrm{M_x} + \mathrm{M_y} >=< \mathrm{S_Z^{(x)}} + \mathrm{S_Z^{(y)}} > . \tag{11}$$

The above equation suggests we may have to study the matrix elements of spin, as well as $\tilde{\mathrm{E}}$ and holonomies, in order to study U(1) invariance. Fortunately, the $\tilde{\mathrm{E}}$ operators are identical to the spin operators, since $\tilde{\mathrm{E}}^i_A$ brings down a factor of $S^{(i)}_A$; we get spin matrix elements with no extra effort.

III Coherent States

This section proposes a set of candidate coherent states. I cannot use the coherent states available in the literature, since these are for the general case, full SU(2) symmetry, and do not exhibit the O(3) symmetry discussed in the previous section. However, the results for full SU(2) motivate a tentative choice of states for the planar

case, together with a choice of parameters for labeling these states. The second paper in this series verifies in detail that the candidate states have the properties required of coherent states.

What properties should a coherent state possess? The Hamiltonian is constructed entirely from spin-half holonomies and $\tilde{\mathrm{E}}$ operators. When these quantum operators act upon the coherent state, the result should be c-numbers. From the previous section, the holonomies should be thought of as spin one spherical harmonics

$$\mathrm{D}^{(1)}_{0\mathrm{M}}(-\phi + \pi/2, \theta/2, \phi - \pi/2) := \mathrm{D}^{(1)}(\mathrm{h})_{0\mathrm{M}},$$

while the $\tilde{\mathrm{E}}$ essentially multiply the state by a spin operator. A set of coherent states should therefore have the properties

$$\begin{aligned}
\mid \ldots\rangle &= \sum_{L,M} D^{(L)}_{0M}(h) c(L,M;\ldots); \\
D^{(1)}(h)_{0A} \mid \ldots\rangle &= < D^{(1)}(h)_{0A} > \mid \ldots\rangle + SC; \\
(\gamma\kappa/2)^{-1}\tilde{\mathrm{E}}^x_A \mid \ldots\rangle &= \sum_{L,M} D^{(L)}_{0N}(h) \langle L,N \mid S_A \mid L,M\rangle c(L,M;\ldots) \\
&= (\gamma\kappa/2)^{-1} < \tilde{\mathrm{E}}^x_A > \mid \ldots\rangle + SC,
\end{aligned} \tag{12}$$

SC denotes small corrections, down by order $1/\sqrt{< L >}$, $< L >$ the peak value of L. Eq. (12) means that the coherent state is an approximate eigenvector of the holonomy and triad, with eigenvalues given by the numbers $< D^{(1)}(h)_{0A} >$ and $< \tilde{\mathrm{E}}^x_A >$. If D(h) depends on angles (θ, ϕ), then $< D(h) >$ depends on angles (α, β) which may be interpreted as the peak values of (θ, ϕ). The dots $(\ldots)$ stand for coherent state labels to be introduced shortly.

There are two kinds of SU(2) coherent states available in the literature. The first type has no sum over L in eq. (12), only a sum over M [36, 37]. These coherent states are too simple for present purposes. If L is fixed, the $\tilde{\mathrm{E}}$ operators can be made classical, but not the holonomies: one must superimpose many L's to get sharply peaked angular values for (θ, ϕ).

The second kind of coherent state was suggested by Hall [30] and elaborated for quantum loop gravity by Thiemann and Winkler [22, 23, 24]. Their results were derived for the general case, full local SU(2) symmetry, whereas the planar case possesses only U(1) gauge symmetry. Simply taking a U(1) limit of the general case does

not work, since the planar limit must possess the O(3) symmetry discussed in the previous section.

However, the Thiemann Winkler coherent states may be understood intuitively as generalizations of the minimal uncertainty states for the free particle. One applies a certain recipe to construct the free particle states, and the same recipe works in the general SU(2) case. Once this intuitive approach is understood, it is straightforward to use the same recipe to generate a set of candidate coherent states for the planar case.

I now review the recipe for constructing a coherent state for the free particle. Start from a wave function which is a delta function.

$$\delta(\mathrm{x}-\mathrm{x}_0) = \int \exp[\mathrm{ik}(\mathrm{x}-\mathrm{x}_0)]\, \mathrm{dk}/2\pi.$$

This wave function is certainly strongly peaked, but it is not normalizable. Also, it is peaked in position, but it should be peaked in both momentum and position. To make the packet normalizable, insert a Gaussian operator $\exp[-p^2/(2\sigma^2)]$. (Choosing the Gaussian form is a "cheat", because we know the answer; but for future reference note that all the eigenvalues k^2 of p^2 must be positive, so that the Gaussian damps for all k.) To produce a peak in momentum, complexify the peak position: $x_0 \rightarrow x_0 + ip_0/\sigma^2$. With these changes, the packet becomes

$$\begin{aligned} N \int & \exp[-p^2/(2\sigma^2)] \exp[ik(x-x_0) + kp_0/\sigma^2]\, dk/2\pi \\ &= N \int \exp[-k^2/(2\sigma^2) + ik(x-x_0) + kp_0/\sigma^2]\, dk/2\pi \\ &\qquad = (N \exp[p_0^2/(2\sigma^2)]/\sqrt{2\pi}) \\ &\qquad\qquad \cdot \exp[ip_0(x-x_0) - (x-x_0)^2\sigma^2/2]. \quad (13) \end{aligned}$$

The last line follows after completing the square on the exponential, and exhibits the characteristic coherent state form.

Note there is not just one state, but a family of coherent states, characterized by the parameter σ. The shape of the wave function is highly sensitive to σ; but the peak values x_0, p_0 are independent of σ, as is the minimal uncertainty relation $\Delta x \Delta p = \hbar/2$. The coherent states constructed below contain a parameter t which is analogous to σ.

Now apply the above recipe to the planar case. The position x becomes the pair of angular variables (θ, ϕ) on the group manifold. The complete set of plane waves become a complete set of spherical harmonics. To construct a delta function in angles,

$$\delta(\theta/2 - \alpha/2)\delta(\phi - \beta)/\sin(\alpha/2)$$

I introduce spherical harmonics $D^{(L)}(u)$ depending on angles (α, β) in the same way that the $D^{(L)}(h)$ depend on (θ, ϕ).

$$\begin{aligned} D^{(L)}(u)_{0M} &= D^{(L)}(-\beta + \pi/2, \alpha/2, \beta - \pi/2)_{0M} \\ &= \sqrt{4\pi/(2L+1)}Y_{LM}(\alpha/2, \beta - \pi/2). \end{aligned} \tag{14}$$

Compare eq. (14) to eq. (8). I can now write the delta function in angle as a sum over spherical harmonics.

$$\delta(\theta/2-\alpha/2)\delta(\phi-\beta)/\sin(\alpha/2) = \sum_{\mathrm{L,M}}((2\mathrm{L}+1)/4\pi)\mathrm{D}^{(\mathrm{L})}(\mathrm{h})_{0\mathrm{M}}\mathrm{D}^{(\mathrm{L})}(\mathrm{u})^*_{0\mathrm{M}}. \tag{15}$$

As discussed in the last section, it is more convenient to use D matrices rather than Y_{LM}'s, but the reader who wishes to exhibit the latter can use eqs. (8) and (14) to replace D's by Y's in eq. (15). The sum then takes on a form which may be more familiar, just $\sum YY^*$. The wave number k in the free particle example corresponds to L in the planar case.

Continue with the recipe: dampen the sum using a Gaussian $\exp[-tL(L+1)/2]$. Complexify by extending the angles in u to complex values, replacing u by a matrix g in the complex extension of O(3). The coherent state has the general form

$$\begin{aligned} \mid u, \vec{p}\rangle &= N\sum_{L,M}((2L+1)/4\pi)\exp[-tL(L+1)/2]D^{(L)}(h)_{0M}D^{(L)}(g)^*_{0M} \\ &= N\sum_{L,M}\cdots D^{(L)}(h)_{0M}D^{(L)}(g\dagger)_{M0}. \end{aligned} \tag{16}$$

A vector $\vec{p}$ is needed to characterize the matrix g, as follows. Every matrix in SL(2,C), the complex extension of SU(2), can be decomposed into a product of a Hermitean matrix times a unitary

matrix ("polar decomposition"; see for example [38]). E. g. for the fundamental representation,

$$\mathrm{g} = \mathrm{Hu} = \exp[\tilde{\sigma} \cdot \tilde{\mathrm{p}}/2]\mathrm{u}.$$

It follows that every matrix in O(3) also has a polar decomposition, obtained by restricting the representations of SU(2) to representations with integer spin.

$$\begin{aligned} g^{(L)} &= \exp[\vec{S} \cdot \vec{p}]u^{(L)} \\ &:= H^{(L)}u^{(L)} \end{aligned} \tag{17}$$

At this point I must choose six input parameters: the three Euler angles which determine the unitary matrix u and the three components of $\vec{p}$, which define the complex extension. By analogy with the free particle case, if u determines the peak angles, then the complex extension $\vec{p}$ determines peak values of the canonically conjugate variable, the angular momentum.

In this paper I use the following choices for these parameters. Restrict $\vec{p}$ to be an arbitrary vector in the XY plane.

$$\mathrm{p}_3 = 0. \tag{18}$$

Restrict the unitary matrix u to be an arbitrary rotation with axis of rotation in the XY plane. The Euler decomposition of $u^{(L)}$ is

$$\begin{aligned} u^{(L)} &= u(-\beta + \pi/2, \alpha, \beta - \pi/2) \\ &:= \exp[i\hat{n} \cdot \vec{S}\alpha/2]; \\ \hat{n} &= (\cos\beta, \sin\beta, 0). \end{aligned} \tag{19}$$

(Compare this definition of u to the corresponding definition of the matrix h, given at eqs. (1) and (2). Both u and h have their axis of rotation in the XY plane.)

These choices are certainly plausible. In the limit of no complexity (in the limit $\vec{p} \rightarrow 0$) the coherent state will reduce to the original delta function and u will become the peak value of h. Since h has its axis of rotation in the XY plane, the peak value should also be a matrix with axis in the XY plane.

Also, the generator $\vec{S} \cdot \vec{p}$ for H is the complexification of the generator $\vec{S} \cdot \hat{n}$ for u. Since $\hat{n}$ lies in the XY plane, the same should be true for $\vec{p}$. In short, since h is a very special matrix (its axis lies in the XY plane) the peak matrix u and its complexification H should also be special.

In paper 2, I calculate the expectation values $< \tilde{\mathrm{E}}^x_A >$ and $< D^{(1)}(h)_{0A} >$ only for the case $p_3 = 0$, and axis of u in the XY plane, eqs. (18) and (19). At the end of section III of paper 2 I state a theorem which describes what happens when u and $\vec{p}$ are replaced by more general choices,

$$(\mathrm{u}, \tilde{\mathrm{p}}) \rightarrow (\tilde{\mathrm{u}}, \tilde{\tilde{\mathrm{p}}}),$$

where $\tilde{p}$ and the axis of $\tilde{u}$ need not be in the XY plane. To keep an already too lengthy paper 2 as short as possible, I have not supplied a proof of the theorem; but the industrious reader can easily do so by using the techniques developed up to that point. Use of the more general values seems to add nothing but complexity.

Since $\vec{p}$ now lies in the XY plane, $\vec{p}$ may be parameterized using a magnitude, p, plus an angle μ. It is convenient for later calculations to take μ to be the angle between $\vec{p}$ and $\hat{n}$, the axis of u.

$$\tilde{\mathrm{p}} = \mathrm{p}[\cos(\beta + \mu), \sin(\beta + \mu), 0]. \tag{20}$$

(From eq. (19), β is the angle between $\hat{n}$ and the X axis.)

When the expectation values $< \tilde{\mathrm{E}}^x_A >$ and $< D^{(1)}(h)_{0A} >$ are calculated in paper 2, they turn out to be perpendicular.

$$0 = \sum_{\mathrm{A}} < \mathrm{D}^{(1)}(\mathrm{h})_{0\mathrm{A}} >< \tilde{\mathrm{E}}^x_A > . \tag{21}$$

There is no constraint like this for the expectation values in the case of general SU(2) symmetry.

The following quantum mechanical analogy is helpful (probably, essential) in clarifying the situation. Consider an electron moving in a central potential and described by polar and azimuthal coordinates $(\theta/2, \phi - \pi/2)$. (The definitions of polar and azimuthal angles are unorthodox but acceptable.) The angular wavefunction for the electron, as well as the coherent state eq. (16), are both superpositions of spherical functions Y_{LM}, or equivalently rotation matrices $D(h)_{0M}$. Therefore eq. (16) can serve as a coherent state describing the angular motion of an electron.

When the three quantities $D^{(1)}(h)_{0A}$ are written out, they turn out to be the three components of the (unit) radius vector of the electron. Therefore $< D^{(1)}(h)_{0A} >$ is the peak value of the unit radius vector. Continuing with the analogy, $< \tilde{\mathrm{E}}^x_A >$ is the peak value of the electron angular momentum vector, since the $\tilde{\mathrm{E}}^x_A$ operator, when acting on a D(h), brings down a factor of spin S_A. Because angular momentum is perpendicular to the orbit plane, these two vectors ($< D^{(1)}(h)_{0A} >$ and $< \tilde{\mathrm{E}}^x_A >$) *must be perpendicular*.

This constraint follows from the symmetry, not the dynamics (it does not matter whether the electron is moving in a hydrogen atom or in a spherical well); therefore the spin network coherent state must obey the same constraint. (The expectation values continue to obey the constraint even when the parameters u and $\vec{p}$ are generalized to parameters $(\tilde{u}, \vec{\tilde{p}})$ not lying in the XY plane.)

IV Form of the Action on a Coherent State

Paper 2 calculates the action of $\tilde{\mathrm{E}}^x_A$ and $D^{(1)}(h)_{0A}$ (equivalently, the holonomy $h^{1/2}$) on the coherent state $\mid u, \vec{p}\rangle$. This section is intended to be an overview of those calculations. It summarizes the main steps in the two calculations, with a minimum of detail.

Let O_A stand for either $\tilde{\mathrm{E}}^x_A$ or $D^{(1)}(h)_{0A}$; both have a vector index A. Then from eqs. (16) and (17),

$$\begin{aligned} O_A \mid u, \vec{p}\rangle &= N \sum_{L,M} ((2L+1)/4\pi) \exp[-tL(L+1)/2] \\ &\quad \cdot [O_A D^{(L)}(h)_{0M}] D^{(L)}(u\dagger H)_{M0} \\ &= N \sum_{L,M,L',M'} ((2L+1)/4\pi) \exp[-tL(L+1)/2] \\ &\quad \cdot D^{(L')}(h)_{0M'} \langle L'M' \mid O_A \mid LM\rangle \\ &\quad \cdot D^{(L)}(u\dagger)_{MN} D^{(L)}(H)_{N0}. \end{aligned} \tag{22}$$

(Neither $\tilde{\mathrm{E}}$ nor h changes the "zero" index on $D^{(L)}(h)_{0M}$.)

Eq. (22) is very general, but hard to interpret. However, since the actions of $\tilde{\mathrm{E}}^x_A$ and $D^{(1)}(h)_{0A}$ on the coherent state yield vectors, it will help to introduce an orthonormal triad of geometrically significant unit vectors and expand the right hand side of eq. (22) in these vectors.

I use the rows of the orthogonal matrix $D^{(1)}(u)$ as the orthonormal triad, since these provide a natural set of geometrically significant vectors. Proof: when $O_A = D^{(1)}(h)_{0A}$ in eq. (22) (when O_A is a holonomy), one expects

$$\begin{aligned} D^{(1)}(h)_{0A} \mid u, \vec{p}\rangle &= < D^{(1)}(h)_{0A} >\mid u, \vec{p}\rangle + SC \\ &= D^{(1)}(u)_{0A} \mid u, \vec{p}\rangle + SC; \\ D^{(1)}(u)_{0A} &= \hat{V}(h = u)_A. \end{aligned}$$

(The last line is eq. (9).) I. e. the zeroth row of the matrix $D^{(1)}(u)$ should be the peak value of the holonomy; and at least this row should occur when I expand the right hand side of eq. (22).

The remaining two rows, $D^{(1)}(u)_{\pm,A}$, are also geometrically significant. $D^{(1)}(u)_{0A} = \hat{V}(u)$ is the image of the Z axis under the rotation u. Since u is a rotation around an axis $\hat{n}$, it follows that $\hat{n}$ is perpendicular to $\hat{V}(u)$. The rows $D^{(1)}(u)_{\pm,A}$ are also perpendicular to $\hat{V}(u) = D^{(1)}(u)_{0A}$, therefore are linear combinations of $\hat{n}$ and the other unit vector perpendicular to $\hat{V}(u)$, namely $\hat{n} \times \hat{V}$. It follows that the three rows of the orthogonal matrix $D^{(1)}(u)$ are linear functions of the geometrically significant vectors $\hat{n}$, $\hat{V}$, and $\hat{n} \times \hat{V}$ $\Box$

The exact relation between these three vectors and the rows of $D^{(1)}(u)$ is worked out in paper 2. For now, note it is not too hard to produce a factor of $D^{(1)}(u)$ in eq. (22). Since the matrix element $< O_A >$ in eq. (22)is essentially a Clebsch-Gordan coefficient, one can use the rotation properties of this coefficient to move the $D^{(L)}(u\dagger)$ matrix through the matrix element and produce factors of $D^{(1)}(u)$.

$$\langle L'M' \mid O_A \mid LM\rangle D^{(L)}(u\dagger)_{MN} = D^{(L')}(u\dagger)_{M'M''}\langle L'M'' \mid O_B \mid LN\rangle D^{(1)}(u)_{BA};$$

$$\begin{aligned} O_A \mid u, \vec{p}\rangle = N \sum_{L,N,L',M'} &((2L+1)/4\pi)\exp[-tL(L+1)/2] \\ \cdot D^{(L')}(hu\dagger)_{0M'}\langle L'M' \mid O_B \mid LN\rangle &D^{(1)}(u)_{BA} D^{(L)}(H)_{N0}. \end{aligned} \tag{23}$$

The $D^{(1)}(u)_{BA}$ can be expressed in terms of $\hat{n}$, $\hat{V}$, and $\hat{n} \times \hat{V}$, to reveal the component along each of these directions.

Next, focus on the sum over L, N, L$'$, M$'$ in eq. (23): how can it be simplified? We wish to prove that the coherent state $\mid u, \vec{p}\rangle$

is an (approximate) *eigenstate* of the operator O_A. This means it is desirable to simplify the sum in eq. (23), until it resembles as much as possible the corresponding sum for the original coherent state. Eq. (23) contains the product $D^{(L')}(hu\dagger)_{0M'}D^{(L)}(H)_{N0}$. The original coherent state, eq. (16), contains a similar product, but with indices M$'$ and N replaced by a single index M, and L$'$ = L.

The indices M$'$ and N in eq. (23) are not necessarily equal. They differ by the index B, because of the selection rules obeyed by the Clebsch-Gordan coefficient in $\langle L'M' \mid O_B \mid LN\rangle$.

$$\mathrm{N} = \mathrm{M}' - \mathrm{B}.$$

Also, when O_A is an $\tilde{\mathrm{E}}$ operator, L$'$ will turn out to be L; but for O_A a holonomy, L and L$'$ will differ by one unit: L$'$ = L $\pm$ 1.

Despite these mismatches, it is possible to rearrange the sum in eq. (23) so that it closely resembles the sum in the original coherent state. The philosophy is to leave the $D(hu\dagger)$ factor alone; this factor is oscillatory, and hard to approximate. Work with the D(H) factor instead, which turns out to be exponential and relatively simple. Rewrite eq. (23) as

$$\begin{aligned} O_A \mid u, \vec{p}\rangle \;\; = \;\; & N \sum_{L,L',M'} ((2L+1)/4\pi) \exp[-tL(L+1)/2] \\ & \cdot D^{(L')}(hu\dagger)_{0M'} D^{(L)}(H)_{M'0} \\ & \cdot [\langle L'M' \mid O_B \mid L, M'-B\rangle D^{(L)}(H)_{M'-B,0}/D^{(L)}(H)_{M',0}] \\ & \cdot D^{(1)}(u)_{BA}. \end{aligned} \tag{24}$$

For O_A an $\tilde{\mathrm{E}}$ operator, L$'$ = L, and the product

$$\mathrm{D^{(L')}(hu\dagger)_{0M'}D^{(L)}(H)_{M'0}}$$

is now identical to the corresponding product in the original coherent state. For O_A a holonomy operator, $L' = L \pm 1$; but it is best not to replace the $D^{(L)}(H)_{M'0}$ factor by $D^{(L')}(H)_{M'0}$. The L$'$ = L $\pm$1 states have peak values of L shifted by $\pm$1 unit. Since this is a real physical effect which cannot be removed by changing the D(H) factors, it is simpler and clearer to leave the second line, eq. (24), as is.

We are not out of the woods yet: the square bracket in eq. (24) is a function of L and M, and cannot be taken out of the sum.

However, the factor $D^{(L)}(H)_{M0}\exp[-tL(L+1)/2]$ turns out to have the normalized Gaussian form typical of coherent states,

$$\begin{aligned} D^{(L)}(H)_{M0}\exp[-tL(L+1)/2] \cong& \\ &\sqrt{t/\pi}\exp[-t(L+1/2-p/t)^2/2] \\ &\sqrt{1/[(L+1/2)\pi]}\exp\{-M^2/[2(L+1/2)]\}f(p,t). \end{aligned}$$

Therefore, to get the dominant contribution, one can expand the square bracket in eq. (24) around the mean values, $< L+1/2 > =$ p/t and $< M > = 0$, and keep the first, constant term in the series. Let b(L, M; L′-L, B) denote the square bracket in eq. (24).

$$\begin{aligned} b(L,M) &= b(< L >, 0) + (< L > \partial_L b)\Delta L/ < L > \\ &\quad +(< L > \partial_M b)\Delta M/ < L > + \cdots ; \\ \Delta L &:= L- < L >= L + 1/2 - p/t; \\ \Delta M &:= M- < M >= M. \end{aligned} \tag{25}$$

I have suppressed the arguments L′ - L and B, since they are held fixed. The expansion coefficients b($< L >$, 0) and $< L > \partial_X b$ all turn out to have the same order of magnitude. Because of the Gaussian nature of D(H), the factors of $\Delta X/ < L >$ are order $\sigma_X/ < L >$, where σ_X is the standard deviation associated with the variable X (X = L or M). The first term, b($< L >$, 0), is therefore the approximate eigenvalue of the operator O_A, while the remaining terms in the series are small corrections (SC) down by factors of $\sigma_X/ < L >$.

The final results for the action of holonomy and triad on the coherent state are (with $h^{(1/2)}$ replaced by $D^{(1)}_{0A}(h)$)

$$\begin{aligned} D^{(1)}_{0A}(h) \mid u, \vec{p}\rangle &= [D^{(1)}_{0A}(u) = \hat{V}_A] \mid u, \vec{p}\rangle + SC; \\ (\gamma\kappa/2)^{-1}\tilde{\mathrm{E}}^x_A \mid u, \vec{p}\rangle &= < L > (\hat{n}_A \cos\mu - (\hat{n}\times\hat{V})_A \sin\mu) \mid u, \vec{p}\rangle + SC. \end{aligned}$$

The expectation value of spin Z component (relevant for U(1) conservation; see eq. (11)) is

$$< \mathrm{S}_\mathrm{Z}^{(\mathrm{x})} + \mathrm{S}_\mathrm{Z}^{(\mathrm{y})} >=< \mathrm{L}_\mathrm{x} > \sin(\alpha_\mathrm{x}/2)\sin\mu_\mathrm{x} + < \mathrm{L}_\mathrm{y} > \sin(\alpha_\mathrm{y}/2)\sin\mu_\mathrm{y}.$$

This is the calculation in broad outline, and paper 2 fills in the details.

A note on the appearance of L+1/2 rather than L in Gaussian expansions of the D(H): the factor L+1/2 is an approximation for

$$\sqrt{\mathrm{L}(\mathrm{L}+1)} = \mathrm{L} + 1/2 + \mathrm{order}(1/\mathrm{L}).$$

This paper considers terms to order $1/\sqrt{L}$ but does not always take into account terms down by 1/L. The factor of 1/2 is therefore an educated guess rather than the result of a calculation. The theory contains a number of scalars (the magnitude of S; standard deviations); and the scalar in SU(2) theory is $\sqrt{L(L+1)}$, rather than L. I use the easy-to-type L+1/2, rather than L, whenever it is likely that a scalar quantity should be $\sqrt{L(L+1)}$, even though my approximations may not be accurate enough to establish this.

V Conclusion

Note two nice features of the calculation sketched in section IV: the use of the rotation properties of the Clebsch-Gordan coefficients at eq. (23) produces a convenient set of unit vectors; the subsequent power series expansion of the square bracket makes it easy to separate leading terms from small corrections, and evaluate the latter if needed. Both these features carry over to other calculations where the coherent states exhibit a continuous symmetry, including the general SU(2) calculation.

The general SU(2) case involves a complete set of functions on a group manifold, whereas the present, O(3) case involves a complete set of harmonics on a sphere. This seems to be a small difference; nevertheless, the O(3) case is unusual in two respects. As shown in section III, the expectation values of $\tilde{\mathrm{E}}$ and holonomy must be perpendicular. Also, if peak values are tuned by means of an input matrix u and vector $\vec{p}$, in the SU(2) case the matrix u always equals the peak value of holonomy h; whereas in the O(3) case we shall see in paper II that u equals the peak value only for the specific parameter value $p_3 = 0$.

I am not clear as to the deeper mathematical reasons for these differences. However, it seems clear that one should proceed with caution when trying to construct coherent states using a basis other than representations on a group manifold.

A Cylindrical Symmetry

What happens when the symmetry is cylindrical, rather than planar? On first inspection, the cylindrical case appears to be a simple relabeling of the planar case. The wave propagates in the r direction in the cylindrical case, and a right-handed coordinate system is desirable. Therefore one can relabel

$$\mathrm{x}, \mathrm{y}, \mathrm{z} \rightarrow \phi, \mathrm{z}, \mathrm{r};\ \mathrm{X}, \mathrm{Y}, \mathrm{Z} \rightarrow \Phi, \mathrm{Z}, \mathrm{R}$$

The gauge fixing goes through as for the planar case [25]. In particular, the $\tilde{\mathrm{E}}$ matrix is again block diagonal. The 1x1 block now contains $\tilde{\mathrm{E}}^r_R$, rather than $\tilde{\mathrm{E}}^z_Z$; the 2x2 block now has rows and columns labeled by (ϕ, z) and (Φ, Z). Similarly, the A^I_i matrix is also block diagonal. One can choose a representation of the generators such that "S_R" is always diagonal (always given by S_Z), while "S_Φ" and "S_Z" are given by S_X and S_Y respectively. Except for the relabeling, the cylindrical theory now looks exactly like the planar theory, suggesting that a solution to the former will also work for the latter.

Can this be right? We know that in field theory the two symmetries have different solutions, because the boundary conditions are different (different falloff with r or z, at r or z $\rightarrow$ infinity; finiteness requirement at the origin, r = 0, for the cylindrical case). However, in a spin network formulation manifold indices, (r, ϕ, z) or (x, y, z), disappear. They are integrated over areas or curves in order to form spatial diffeomorphism invariants; and cylindrical and planar coordinates differ precisely by a spatial diffeomorphism. Somehow, in our haste to get rid of the coordinates on the manifold (ugly, because arbitrary), we have lost the distinction between cylindrical and planar. We still have the local Lorentz indices (R, Φ, Z) or (X, Y, Z); but these are too local to reveal the difference between the two symmetries.

Perhaps the solution is to recover some global properties in a spin network context? Yes, but hiding the manifold coordinates was a big step forward, and one would like to avoid a step backward. It should be possible to introduce global properties without going back to the underlying manifold coordinates. E. g., for the cylindrical case, one must impose finiteness of the total energy within a unit length of cylinder; and for the planar case, finite total energy per unit area. Since the energy is given by the integral of a surface term at spatial

infinity, in field theory these finiteness requirements imply different asymptotic behaviors for cylindrical and planar fields.

In spin network theory, one can require a finite total energy, without going to the underlying field theory or reintroducing manifold coordinates. The spin network is only superficially one-dimensional. Since both transverse holonomies are integrated over an unspecified repeat distance, the spin network surface term will be two dimensional, an integral over the two transverse repeat distances. It is not clear how large this transverse area is, but one can divide the energy integral by a transverse area integral,

$$\int \tilde{\mathrm{E}}^z_Z \, \mathrm{dx}\, \mathrm{dy} \text{ or } \int \tilde{\mathrm{E}}^r_R \, \mathrm{d}\phi \, \mathrm{dz}.$$

The resultant energy per unit area is independent of the unknown repeat distances.

(I have imagined the transverse repeat distances as arbitrary, but fixed during the calculation. Alternatively, one can assume the symmetric theory is the limit as the repeat distance is taken to zero [39]. I am not sure which point of view is correct, or whether it matters. However, both points of view would seem to require the area integral in the denominator.

In field theory, the linearly polarized cylindrical case is especially interesting because it can be solved exactly [40]. In the spin network approach, however, it may not be straightforward to impose the constraints which reduce the wave to a single linear polarization [27]. The field theoretic constraints set an $\tilde{\mathrm{E}}$ and its conjugate momentum equal to zero. In the spin network approach, the conjugate momentum, a component of the extrinsic curvature, is constructed by commuting the Euclidean Hamiltonian with the volume operator. Since the spin network Hamiltonian is non-local (since it involves holonomic loops connecting two or more vertices), the conjugate momentum becomes non-local.

Furthermore, in order to solve the field theoretic case, one must (specialize to linear polarization, then) canonically transform to a new coordinate system: R, T coordinates, in the notation of Kuchař [40]. The new T coordinate is a function of a momentum which Kuchař calls π_γ. In the notation of the present paper, that momentum is

$$\Sigma' \, \mathrm{K}^A_a \, \tilde{\mathrm{E}}^a_A.$$

K is the extrinsic curvature. The prime on the summation means omit r and R in the sum over indices. Again, this operator is not defined in a spin network context (or in quantum field theory for that matter); again, it must be defined using commutators of volume with Hamiltonian; again, it is non-local.

This is as far as one can go without a specific Hamiltonian. In the field theoretic case, the linear polarization constraints may be used to simplify the Hamiltonian. Despite the non-locality, A similar simplification may occur in the spin network case.

References

[1] A. Ashtekar, New Hamiltonian formulation of general relativity, Phys. Rev. D **36** (1987) 1587

[2] A. Sen, Gravity as a spin system, Phys. Lett. **119**B (1982) 89

[3] F. Barbero, Real Ashtekar variables for Lorentzian signature spacetimes, Phys. Rev. D **51** (1995) 5507; Phys. Rev. D **51** (1995) 5498.

[4] T. Thiemann, Quantum spin dynamics (QSD), Class. Quantum gravity, **15** (1998) 1839 [gr-qc/9606089]

[5] T. Jacobson and L. Smolin, Nonperturbative quantum geometries, Nucl. Phys. B**299** (1988) 295

[6] C. Rovelli and L.Smolin, Knot theory and quantum gravity, Phys. Rev. Lett. **61** (1998) 1155; Loop space representation for quantum general relativity, Nucl. Phys. B**331** (1990) 80

[7] R. Gambini and J.Pullin, Nonstandard optics from quantum space-time, Phys. Rev. D**59** (1999) 12402

[8] J. Alfaro, H.A. Morales-Tecotl, and L.F. Urrutia, Loop quantum gravity and light propagation, Phys. Rev. D**65** (2002) 103509

[9] S. Liberati, T. Jackson, and D. Mattingly, Lorentz violation at high energy, Annals of Physics **321** (2006) 150-96

[10] R. Bluhm, Overview of the standard model extension: implications and phenomenology of Lorentz violation, in *Special Relativity. Will it survive the next 101 years?* edited by J. Ehlers and C. Lammerzahl(Springer, Berlin 2006)

[11] C. Rovelli and L. Smolin, Discreteness of area and volume in quantum gravity, Nucl. Phys. B**442** (1995) 593; erratum, Nucl. Phys. B**456** (1995) 734

[12] L.G. Garay, Quantum gravity and minimum length, Int. J. Mod. Phys. A**10** (1995) 145-66 [gr-qc/ 9403008]

[13] M. Bojowald, Isotropic loop quantum cosmology, Class. Quantum Grav. **19** (2002) 2717 [gr-qc/0202077]; Homogeneous loop quantum cosmology, Class. Quantum Grav. **20** (2003) 2595 [gr-qc/0303073].

[14] C. Rovelli, Black hole entropy from loop quantum gravity, Phys. Rev. Lett. **14** (1996)

[15] K. Krasnov, Geometrical entropy from loop quantum gravity, Phys. Rev. D**55** (1997) 3505

[16] L. Smolin, The classical limit and the form of the hamiltonian constraint in non-perturbative quantum general relativity [gr-qc/9609034].

[17] D. E. Neville, Long range correlations in quantum gravity, Phys. Rev. D **57** (1999) 044032

[18] E. Alesci and C. Rovelli, The complete loop quantum gravity propagator: I. Difficulties with the Barrett- Crane Vertex, arXiv [gr-qc] 0708.0883v1

[19] J. Ambjorn, J. Jurkiewicz, and R. Loll, Phys. Rev. Lett. **93** (2004) 131301

[20] F. Markopoulou, Coarse graining in spin foam models, Class. Quantum Grav. **20** (2003) 777 [gr-qc/ 0203036]

[21] R. Oeckl, Renormalization of discrete models without background, Nucl. Phys. B**657** (2003) 107 [gr-qc/ 0212047]

[22] T. Thiemann, Gauge field coherent states (GCS):I. General properties, Class. Quantum Grav. **18** (2001) 2025-64 [hep-th/0005233]

[23] T. Thiemann and O. Winkler, Gauge field coherent states II (GCS): Peakedness properties, Class. Quantum Grav., **18** (2001) 2561 [hep-th/0005237]

[24] T. Thiemann and O. Winkler, Gauge field coherent states III (GCS): Ehrenfest theorems, Class. Quantum Grav., **18** (2001) 4629 [hep-th/0005234]

[25] V. Husain and L. Smolin, Exactly Solvable Quantum Cosmologies from Two Killing Field Reductions of General Relativity, Nucl. Phys. B **327** (1989) 205

[26] D. E. Neville, Volume operator for spin networks with planar or cylindrical symmetry, Phys. Rev. D**73** (2006) 124004 [gr-qc/0511005]; erratum (to be published)

[27] D.E.Neville Volume operator for singly polarized gravity waves with planar or cylindrical symmetry, Phys. Rev. D**73** (2006) 124005 [gr-qc/0511006]

[28] D. E. Neville, Planar Spin Network Coherent States II. Matrix Elements.

[29] D. E. Neville, On the Classical Limit of Spin Network Gravity: Two Conjectures.

[30] B.T. Hall, Phase space bounds for quantum mechanics on a compact Lie group, Commun. Math. Phys. **184** (1997) 233-50

[31] C. Rovelli, The basis of the Ponzano-Regge-Turaev-Viro-Ooguri quantum gravity model in the loop representation basis, Phys. Rev. D **48** (1993) 2702-07 [hep-th/9304164]

[32] M. Arnsdorf, Relating covariant and canonical approaches in triangulated models of quantum gravity, Class. Quantum Grav. **19** (2002) 1065-91

[33] A. Perez, Spin Foam Models for Quantum Gravity, Class. Quantum Grav. **20** (2003) R43 [gr-qc/0301113 v2]

[34] M. Gaul and C. Rovelli, A generalized Hamiltonian constraint operator in loop quantum gravity and its simplest Euclidean matrix elements, Class. Quantum Grav. **18** (2001) 1593-1624 [gr-qc/0011106]

[35] B. Bahr and T. Thiemann, Gauge Invariant Coherent States for Loop Quantum Gravity I: Abelian groups ArXiv [gr-qc]0709.4619; Gauge Invariant Coherent States for Loop Quantum Gravity II: ArXiv [gr-qc]0709.4636

[36] J.M. Radcliffe, Some properties of coherent spin states, J. Phys. A:Gen. Phys. **4** (1971) 313-23

[37] A. Perolomov, *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin 1986)

[38] B.G. Hall, *Lie Groups, Lie Algebras, and Representations: an Elementary Introduction* (Springer-Verlag, New York 2003)

[39] M. Bojowald, *Quantum Geometry and Symmetry*, (Shaker Verlag, Aachen, 2000)

[40] K. Kuchař, Canonical Quantization of Cylindrical Gravitational Waves, Phys. Rev. D **4** (1971) 955